From the defocusing nonlinear Schroedinger to the complex Ginzburg-Landau equation

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Perturbation approaches developed so far for the dark soliton solutions of the (fully integrable) defocusing nonlinear Schroedinger equation cannot describe the dynamics resulting from dissipative perturbations of the Ginzburg-Landau type. Here spatially slowly decaying changes of the background wavenumber occur which requires the use of matching technics. It is shown how the perturbation selects a 1 or 2-parameter subfamily from the 3-parameter family of dark solitons of the nonlinear Schroedinger equation. The dynamics of the perturbed system can then be described analytically as motion within this selected subfamily yielding interesting scenarios. Interaction with shocks occurring in the complex Ginzburg-Landau equation can be included in a straight forward way.

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I. INTRODUCTION

The defocusing non-linear Schroedinger equation (NLSE)

$$\partial_t A = i \left(\partial_x^2 - |A|^2 \right) A \tag{1}$$

has a 3-parameter family of dark solitons of the form $\mathcal{A} = \mathcal{A}_0(x - vt) \exp(-i\Omega t)$. In contrast to the pulse-like bright solitons of the focusing case dark solitons have a hole-like shape. In their center they have a strong depression of the modulus which decays into plane-wave states for $\zeta := x - vt \rightarrow \pm\infty$. For arbitrary but fixed $\Omega > 0$ and Q in a range such that $\kappa := \sqrt{(\Omega - 3Q^2)/2}$ is positive one can derive systematically the two-parameter family of standing soliton solutions $(v = 0)$

$$\mathcal{A}_{Q,\Omega}(x,t) = \sqrt{2} \left[\kappa \tanh(\kappa x) - iQ \right] \exp(iQx - i\Omega t) . \tag{2}$$

From Galilean invariance of the NLSE one then obtains the 3-parameter family of soliton solutions moving with constant velocity v $(\zeta = x - vt)$:

$$\mathcal{A}_{k,v,\Omega}(\zeta,t) = \mathcal{A}_{Q,\bar{\Omega}}(\zeta) \exp\left[i(\frac{v}{2}\zeta + \frac{v^2}{4}t) \right] \tag{3}$$

where $\bar{\Omega} = \Omega + \frac{v^2}{4}$. They have the asymptotic wave number $k = Q + \frac{v}{2}$.

The aim of this work is to describe the dynamics resulting from a dissipative perturbation of Ginzburg-Landau type largely as a motion within the three dimensional soliton space spanned by (k, v, Ω). Thus we take the family parameters as time-dependent (slow) variables while other degrees of freedom follow adiabatically. The problem here is that −in contrast to pulses (where this approach is well known [1])− dark solitons do not decay for large $|\zeta|$ and the wavenumber k of the asymptotic plane-wave states cannot be altered globally in an infinite system. Consequently, to our knowledge, only perturbation methods for situations with unchanging k have been developed so far (see e.g. [2] and references therein). This restriction is, however, not applicable to describe the phenomena presented below for Ginzburg-Landau type perturbations. Simulations show that $k = k(t)$ becomes

time dependent in a <u>finite</u> region around the soliton center (inner region). In this region (of size $|(\partial_t k)\zeta| \ll 1$, see eq.(26) of [5]) the system remains for all times close to one of the (k, v, Ω)-solitons. The corresponding <u>global</u> solution has to be constructed by asymptotic matching. Here, however, in order to describe the approximate dynamics of the soliton center we can restrict our calculations to the inner region without performing the matching procedure explicitly. In order to ensure that a global solution exists we will make use of the result formulated in [5] (section II.C) that any local solution which does not diverge exponentially for $\kappa|\zeta| \gg 1$ (outer region) can be matched to a global one. This provides the boundary conditions for A in the inner region which for small values of $\partial_t k$ overlaps with the outer region. In particular the procedure allows to deal with slowly (algebraically) diverging terms which arise in the inner region from the change of the wavenumber k.

Adding perturbations of Ginzburg-Landau type to the NLSE one obtains a complex Ginzburg-Landau equation (CGLE)

$$\partial_t A = i(\partial_x^2 - |A|^2)A + (r + \beta\partial_x^2 - \gamma|A|^2 + \delta|A|^4)A \tag{4}$$

where we wish to consider $r, \beta, \gamma > 0$ and $|\delta| \ll \gamma$. It has three parameters that cannot be scaled away, e.g. β, γ and δ. Then r depends on the choice of scale [†]. In general δ may be complex, but for simplicity we here assume it to be real. The scenarios we are going to describe occur in a parameter regime where $\gamma \sim O(\beta^2)$. However, since terms containing β^2 may not contribute to the calculations due to PT-symmetry (parity and time reversal) of the NLSE for constantly moving solutions (only odd powers of the perturbation are allowed) it will be consistent to treat γ together with β in a first-order perturbation calculus.

Although we assume $|\delta| \ll \gamma$, which is indeed relevant for many systems, δ may play a crucial role for the <u>qualitative</u> dynamics of the equation as was shown in [3]. For $\delta \equiv 0$ a 1-parameter subfamily, the so-called Nozaki-Bekki (N.B.) holes, is selected from the dark solitons. The analytical form of this subfamily was presented first by Nozaki and Bekki [4] (see e.g. section I.A of [5] for the explicit form of these solutions). They were found to be stable in a belt-like region in parameter space which is limited by the instability of the asymptotic plane-wave states on one side and the instability of the core on the other side. In the NLS limit the core of the standing hole $(v = 0)$ is stable for $\gamma < (4/3)\beta^2$ as can be seen from the calculations presented below.

For $\delta \neq 0$, however, this subfamily is destroyed everywhere in parameter space leaving only the standing $(v = 0)$ symmetric solution. More precisely simulations show that a N.B. hole which would be stable for $\delta = 0$ is either accelerated $(\delta < 0)$ or decelerated $(\delta > 0)$. So for $\delta > 0$ the standing hole persists while for $\delta < 0$ it is unstable against acceleration. $\delta = 0$ is thus the threshold of a stationary bifurcation. In the limit of small velocities the scenario can be described by the phenomenological equation

$$\dot{v} = \lambda_\delta \delta v \tag{5}$$

where we have used v to parametrize the 1-dimensional subspace of (k, v, Ω) in which the dynamics takes place. (The dot refers to the time derivative.) One issue of this work is to determine the phenomenological constant λ_δ analytically.

The scenario becomes richer when (with $\delta > 0$) we cross the line in parameter space where the core of the standing hole loses its stability. Then the system undergoes a Hopf bifurcation and the center of the holes start to oscillate in space and time. To describe this oscillatory behavior one needs two coupled modes. As normal form of the bifurcation one obtains for small velocities and small u

$$\begin{aligned} \dot{u} &= (\lambda - gv^2)u + \delta_1 v \\ \dot{v} &= \mu u + \delta_2 v \end{aligned} \tag{6}$$

where u refers to another (1-parameter) subspace of (k, v, Ω) which will be determined. The other symbols in eq.(6) refer to constant coefficients with δ_1 and δ_2 of order δ. For large negative λ $(|\lambda| \gg |\delta|)$ eq.(5) is recovered

with $\lambda_\delta = \delta_2 - \mu\delta_1/\lambda$. With the perturbation method presented in the following it is in principle possible to determine all these phenomenological coefficients quantitatively. For simplicity we will restrict ourselves to the linear part of eqs.(6).

II. DESTRUCTION OF THE NOZAKI-BEKKI FAMILY

We start with an analytic proof that the 3-parameter soliton family is indeed destroyed if $\delta \neq 0$. From the asymptotic states (large $|\zeta|$) alone one already finds that not more than a 1-parameter subfamily can possibly survive the Ginzburg-Landau type perturbations. The selection criteria for this subfamily are

i) The CGLE has a dispersion relation $\omega = \omega(q) = r/\gamma + (1 - \beta/\gamma)q^2 + \delta/\gamma(r/\gamma - \beta/\gamma q^2)^2 + O(\delta^2)$ for plane waves with wave number q. (The frequency Ω in the co-moving frame is related to the frequency ω in the laboratory frame via $\Omega = \Omega(q,v) = \omega(q) - vq$.) Thus Ω is completely determined by the asymptotic soliton wave number k and the velocity.

ii) Phase conservation becomes non trivial and relates the asymptotic wave numbers to the velocity.

To understand this second point one has to realize that the perturbation alters the symmetry of the solitons in a way that the asymptotic wave number q_1 for $\zeta \ll 1$ is no longer equal to that for $\zeta \gg 1$ (q_2). Then phase conservation requires that both sides rotate with the same frequency $\Omega = \omega(q_i) - vq_i$ ($i = 1, 2$) in the co-moving frame. This yields

$$v = \frac{\omega(q_1) - \omega(q_2)}{q_1 - q_2} = 2(1 - \frac{\beta}{\gamma})k - 2\delta\frac{\beta}{\gamma^2}(2\frac{r}{\gamma} - \frac{\beta}{\gamma}(q_1^2 + q_2^2))k + O(\delta^2) \tag{7}$$

where we defined $k := \frac{1}{2}(q_1 + q_2)$. Below we will use k ($k \propto v + O(\dot{k})$) to label the selected soliton family $\mathcal{A}_s[k]$.

To see whether the remaining 1-parameter subspace actually survives we consider the CGLE eq.(4) in the frame moving with velocity v ($\zeta = x - vt$) and rotating with frequency Ω. For later convenience we multiply the equation by $(\beta - i)$. Separating the resulting equation into an unperturbed part $F[A]$ and the perturbation $I[A] - i\partial_t A + O(\beta^2)$ one obtains by neglecting terms of order β^2

$$F[A] := (\partial_\zeta^2 + iv\partial_\zeta + \Omega - |A|^2)A \tag{8}$$
$$= i\left[(r - \beta\Omega) - (\gamma - \beta)|A|^2 + \delta|A|^4 + i\beta v\partial_\zeta - \partial_t\right]A := I[A] - i\partial_t A$$

where we assumed $\partial_t A$ to be of order β^1 in the co-moving frame. (As noted above terms containing even powers of the perturbation do not contribute to our calculations due to symmetry. So the next relevant corrections to eq.(8) are actually of order β^3.) Now we make the ansatz

$$A(\zeta, t) = \mathcal{A}(\zeta, t) + W(\zeta) \tag{9}$$

where – in the spirit of the matching approach – $W(\zeta)$ is of order β^1 in a finite region around the hole center. $\mathcal{A}$ refers to one of the soliton solutions eq.(3) with given (k, v, Ω). At linear order in β one obtains the inhomogeneous linear differential equation (valid in the inner region)

$$LW = I[\mathcal{A}] - i\partial_t A \tag{10}$$

with

$$LW := \frac{\delta F}{\delta A}[\mathcal{A}]W + \frac{\delta F}{\delta A^*}[\mathcal{A}]W^* \tag{11}$$

for any function $W(\zeta)$. (The time derivative $\partial_t A$ in eq.(10) – which has to be zero in the stationary case considered here first – will be important in the next subsections.) Note that L is selfadjoint. From its neutral

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modes $i\mathcal{A}$ and $\partial_\zeta \mathcal{A}$, which can be derived from gauge (or 'phase') and translational invariance of the NLSE, respectively one can construct a <u>localized</u> mode $\Psi_{loc} = \partial_\zeta \mathcal{A} - ki\mathcal{A}$, which decays exponentially for large $|\zeta|$. Fredholms alternative then yields the solvability condition for eq.(10) (with boundary condition that W should not diverge exponentially in space)

$$\int_{-\infty}^{+\infty} d\zeta\, Re\left((I[\mathcal{A}] - i\partial_t A)\Psi_{loc}^*\right) = 0 \tag{12}$$

which contains no boundary terms. So in principle the destruction of the hole family by the higher-order perturbation can be demonstrated immediately by inserting the solutions (3) (combined with eq.(7) and the dispersion relation) into this equation. In Appendix A we derive the identity

$$Re\left(I[\mathcal{A}]\Psi_{loc}^*\right) = -\delta k \frac{\beta}{\gamma}(\mathcal{R}^2 - \mathcal{R}_\infty^2)(\mathcal{R}^4 - \mathcal{R}_\infty^4) - \frac{\gamma}{2}\Delta v(\mathcal{R}_\infty^2 - \mathcal{R}^2)^2 = \delta k \frac{\beta}{\gamma}(\mathcal{R}_\infty^2 - \mathcal{R}^2)^3 + O(\delta^2) \tag{13}$$

(with $\mathcal{R} := |\mathcal{A}|$ and $\mathcal{R}_\infty := \mathcal{R}(\zeta = \infty) = \Omega + vk - k^2$) where

$$\Delta v := v - 2(1 - \beta/\gamma)k \sim O(\delta) \tag{14}$$

(compare eq.(7)) has been defined for later convenience. To derive the rhs of eq.(13) we have made use of the phase-conservation equation (7) which yields in the NLS limit $\Delta v = -4\delta k(\beta/\gamma^2)\mathcal{R}_\infty + O(\delta^2)$. In the next subsection where perturbations of the asymptotic plane-wave states will be taken into account explicitly this expression will have to be generalized. Since $\mathcal{R}^2 < \mathcal{R}_\infty^2$ for finite ζ one immediately sees from the rhs of eq.(13) that the solvability condition, eq.(12) with $\partial_t A = 0$, is identical with the condition $\delta = 0$ (unless $k = 0$ and thus $v = 0$).

III. ACCELERATION INSTABILITY

In the last section we saw that the perturbed cubic CGLE ($\delta \neq 0$) has no quasi stationary solutions with $v \neq 0$ in the vicinity of the defocusing NLS equation. The next step now is to make an ansatz for a weakly time-dependent solution with a free parameter that can be determined by condition (12). At values of β and γ where (for $\delta = 0$) the selected subfamily (or, more precisely, the solutions presented by Nozaki and Bekki) is stable we observed numerically that for $0 \neq |\delta| \ll \gamma$ the system moves in the vicinity of this 1-parameter family $\mathcal{A}_s = \mathcal{A}_s[k]$ with $\Delta v = O(\delta)$ and $\Omega = \Omega(k)\,[+O(\delta)]$. We thus take the family parameter k (alternatively one could use $v \propto k + O(\dot{k})$ as family parameter) as slow variable, assuming that the other degrees of freedom follow adiabatically. This leads to the ansatz:

$$\partial_t A = \dot{k}\partial_k \mathcal{A}_s[k(t)] + O(\delta\beta^3, \delta^2) = i\dot{k}(\zeta\mathcal{A}_s - \frac{\beta}{\gamma}\sqrt{2}) + O(\delta\beta^3, \delta^2, v^2) \tag{15}$$

(where we used $Q = \frac{\beta}{\gamma}k - \frac{\Delta v}{2} + O(\delta)$) valid for a finite region around the soliton center (inner region). For simplicity we restrict our calculations to the limit of small velocities (i.e. small k) which is of major interest. It is straight forward (though lengthy) to generalize them for arbitrary velocities.

Using eq.(15) the integrand of eq.(12) obtains a supplementary term proportional to $\dot{k}$. One has

$$Re\left((I[\mathcal{A}_s] - i\partial_t A)\Psi_{loc}^*\right) = \delta k \frac{\beta}{\gamma}(\mathcal{R}^2 - \mathcal{R}_\infty^2)(\mathcal{R}^4 - \mathcal{R}_\infty^4) - \frac{\gamma}{2}\Delta v(\mathcal{R}_\infty^2 - \mathcal{R}^2)^2 + \dot{k}\left(\frac{\zeta}{2}\partial_\zeta \mathcal{R}^2 - \frac{\beta}{\gamma}\sqrt{2}Re(\Psi_{loc})\right) . \tag{16}$$

Performing the integration then yields

$$-\delta\frac{8}{5}\frac{\beta}{\gamma}\mathcal{R}_\infty^4\, k - \frac{2}{3}\gamma\mathcal{R}_\infty^2\,\Delta v + (1 - 2\frac{\beta}{\gamma})\dot{k} = 0 . \tag{17}$$

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Here Δv has to be determined such that phase conservation is insured. While in the time-independent case phase conservation was given by eq.(7) here this equation has to be altered by including terms with $\dot{k}$ resulting from the time dependence (eq.(15)). In Appendix B we derive a generalization of equation (7) valid for arbitrary β and γ. In the NLS limit it yields

$$\Delta v = -4\delta\mathcal{R}_\infty^2 \frac{\beta}{\gamma^2} k + \left[-h_s + \frac{\beta}{\gamma^2} - \frac{\sqrt{2}\mathcal{R}_\infty}{q_0} \frac{\beta}{\gamma} \right] \frac{\dot{k}}{\mathcal{R}_\infty^2} + O(v^2) \tag{18}$$

$$\text{with} \quad h_s := \frac{\mathcal{R}_\infty^2 - 2\frac{\beta}{\gamma}q_0^2}{2(\beta-\gamma)q_0^2} \simeq \frac{9}{4(\beta-\gamma)^3} - \frac{\beta}{\gamma(\beta-\gamma)}$$

where $q_0 := q_2(v=0) = -q_1(v=0)$ is the asymptotic wave number of the perturbed standing soliton. In the NLS limit one has $q_0 = (\sqrt{2}/3)(\gamma-\beta)\mathcal{R}_\infty + O(\beta^2, \delta)$ as can be obtained either from the Nozaki-Bekki solutions, or, as we show in Appendix B, by integrating eq.(A1). Inserting eq.(18) into eq.(17) yields

$$\lambda_\delta := \lim_{k\to 0} \frac{\dot{k}}{k\delta} = \frac{16}{15} \left[\frac{8}{3} - \frac{2}{3}\frac{\gamma^2}{\beta} h_s + \frac{2\gamma}{\beta-\gamma} - \frac{\gamma}{\beta} \right]^{-1} \Omega^2 \tag{19}$$

where we inserted $\mathcal{R}_\infty^2 = \Omega$ which can be obtained from eq.(3) for the standing hole (with $\Delta v = 0$). At $\gamma = (4/3)\beta^2(1 + O(\gamma/\beta))$ the expression for λ_δ has a pole. (Including terms of $O(\gamma/\beta)$ – which are neglected for simplicity only – actually improves the accuracy of the results, since terms with β^2 may not occur.) As was explained in section III.A of [5] the pole of λ_δ indicates the position of the core-instability threshold so that in the limit $\gamma \sim \beta^2 \to 0$ (for $\delta = 0$) the core of the standing hole is unstable if $\gamma > (4/3)\beta^2(1 + O(\gamma/\beta))$. Literally speaking here the hole is accelerated even for $\delta = 0$.

IV. HOPF BIFURCATION

In the last section we described the acceleration caused by the quintic term ($\delta \neq 0$) as a motion within the one-parameter subfamily of the three-parameter family of dark solitons which is selected under a perturbation of Ginzburg-Landau type. This description loses its validity as one approaches the threshold of the core instability. At the pole of λ_δ (eq.(19)) the prefactor of $\dot{k}$ vanishes so that one has to include higher-order terms (which contain $\ddot{k}$ and $\Delta\dot{v}$) in order to balance the contributions proportional to δ. This leads to the two-mode scenario described by eqs.(6).

We thus extend eq.(15) for the time dependence by writing

$$\partial_t A = \left(\dot{k}\partial_k + \Delta\dot{v}\partial_{\Delta v} \right) \mathcal{A}[k, \Delta v = 0, \Omega(k)] + \ddot{k}\partial_{\dot{k}} W_1 + O(\delta\beta^2, \delta^2) \tag{20}$$

$$= i\dot{k}(\zeta\mathcal{A} - \frac{\beta}{\gamma}\sqrt{2}) + i\frac{\Delta\dot{v}}{\sqrt{2}} + \ddot{k}\partial_{\dot{k}} W_1 + O(\delta\beta^2, v^2)$$

where W_1 is the solution of the equation

$$L W_1 = I_0[\mathcal{A}_s] - i\dot{k}\partial_k \mathcal{A}_s \tag{21}$$

($I_0[\,]$ refers to $I[\,]$ with $\delta = 0$), which can be solved for arbitrary $\dot{k}$ at threshold of the core instability (i.e. at the pole of λ_δ).

At first sight one would assume that explicit knowledge of the perturbation W_1 is required in order to exploit the ansatz (20). However, to calculate the phase-conservation condition related to eq.(20) one only needs the asymptotic form of W_1 (for $\kappa|\zeta| \gg 1$) which has already been calculated (Appendix B) when deriving eq.(18). Continuing the expansion which led to eq.(18 up to the next order one obtains (see Appendix B)

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$$\Delta v = -4\delta\mathcal{R}_\infty^2 \frac{\beta}{\gamma^2}\,k + \left[-h_s + \frac{\beta}{\gamma^2} - \frac{\sqrt{2}\mathcal{R}_\infty}{q_0}\frac{\beta}{\gamma}\right]\frac{\dot{k}}{\mathcal{R}_\infty^2} + \frac{\sqrt{2}\mathcal{R}_\infty}{q_0}\Delta\dot{v} - \left(\frac{2}{\gamma^2} + \frac{9}{8(\gamma-\beta)^4}\right)\left(\frac{\beta}{\gamma} - \gamma h_s\right)\frac{\ddot{k}}{\mathcal{R}_\infty^4}\,, \quad (22)$$

which differs from eq.(18) by the two last terms on the rhs. (Now, however, an explicit result can be given only for the NLS limit since the second mode labeled by Δv is only known in this limiting case.)

In order to exploit the solvability condition (12) we only need to know the integral over $Re[\partial_{\dot{k}}W_1 * \Psi_{loc}^*]$. Symmetry considerations show that it is an odd function of ζ so that the integral vanishes. To see this we consider the symmetry operator $\Pi[Z(\zeta)] = Z(-\zeta)^*$ (for every complex function $Z(\zeta)$). Due to PT-symmetry $F[]$, and thus L, commute, whereas $I[]$ anticommutes with $\Pi[]$. Soliton solutions are eigenfunctions of $\Pi[]$ with eigenvalues $+1$ while the neutral modes of L have the eigenvalue -1. From eq.(21) one finds that W_1 must have eigenvalue -1 and $Re[\partial_{\dot{k}}W_1 * \Psi_{loc}^*]$ is indeed an odd function. So there appears no contribution containing $\ddot{k}$ in the solvability condition eq.(12) and thus eq.(17) has to be extended only by a term proportional to $\Delta\dot{v}$. Inserting eq.(20) into eq.(12) one finds

$$-\delta\frac{8}{5}\frac{\beta}{\gamma}\mathcal{R}_\infty^4\,k - \frac{2}{3}\gamma\mathcal{R}_\infty^2\,\Delta v + (1 - 2\frac{\beta}{\gamma})\dot{k} + \Delta\dot{v} = 0\,. \quad (23)$$

Differentiating equation (23) with respect to time yields

$$\ddot{k} = \frac{2\gamma^2}{3(\gamma - 2\beta)}\mathcal{R}_\infty^2\Delta\dot{v} + o(\delta) \quad (24)$$

(where $o(\delta)$ contains terms containing $\partial_t^3 k$ and $\delta\partial_t k$ which are of order higher than δ^1). This can be used to eliminate $\ddot{k}$ from eq.(22). Solving eqs.(22) and (23) for $\dot{k}$ and $\Delta\dot{v}$ one then obtains a set of equations which can be identified with eqs.(6) by writing $v \propto k + O(\dot{k})$ and $u \propto \Delta v$. It is thus straight forward to extract the somewhat lengthy expressions for the (linear) coefficients of eqs.(6) from eqs.(23) and (22). (Nonlinear coefficients can be obtained in principle by the same method. Calculation, however, are increased considerably.) We give the expression for the frequency $\Omega_{osc} = \sqrt{\delta_1\gamma - (\lambda - \delta_2)^2/4} \simeq \sqrt{\delta_1\gamma}$ of the harmonic oscillations which eqs.(6) describe in the limit $0 < \lambda \ll \delta^{0.5}$. From eqs.(23) and (22) one obtains:

$$\sqrt{\delta_1\gamma} = \frac{\sqrt{\left[\frac{8}{5}\frac{\beta}{\gamma}h_2 - 4\frac{\beta}{\gamma^2}(1 - 2\frac{\beta}{\gamma})\right]\left(1 - \frac{2}{3}\gamma h_1\right)}}{h_2 - (1 - 2\frac{\beta}{\gamma})h_1}\Omega\sqrt{\mathcal{R}_\infty^2\delta} \quad (25)$$

$$\text{with} \qquad h_1 = \frac{3}{\gamma - \beta} - \frac{2\gamma^2}{3(\gamma - 2\beta)}\left(\frac{2}{\gamma^2} + \frac{9}{8(\beta - \gamma)^4}\right)\left(\frac{\beta}{\gamma} - \gamma h_s\right) \quad (26)$$

$$h_2 = \left[-h_s + \frac{\beta}{\gamma^2} - \frac{3}{\gamma - \beta}\frac{\beta}{\gamma}\right] \quad (27)$$

Using the scale $r = \gamma$ this yields $\omega_{osc}/\sqrt{\delta} \simeq 2,58$ for $\beta = 0.123$ and $\gamma = 1.66 * 10^{-2}$. From simulations of the CGLE we found $\omega_{osc}/\sqrt{\delta} = 2.5 \pm 0.01$ (numerical error) yielding reasonable agreement in view of nonlinear effects. For small β and γ simulations become increasingly time consuming since the interaction distance between holes and boundaries or sinks increases (c.f. next section) so that one needs much longer systems in order to avoid boundary effects.

V. INTERACTION OF HOLES AND SHOCKS

In systems which contain more than one hole neighboring holes are generally separated by shocks. While holes which are sources (group velocity and thus causality points outward) and therefore determine largely the asymptotic wave states shocks are sinks and thus behave in a rather passive way. In simulations shocks moving

with constant velocities are formed when propagating waves with different wave numbers collide. The velocity of a shock is then determined by the incoming wave numbers via eq.(7), which holds for any localized object in the CGLE.

In the presence of a shock the above calculations have to be altered. If the shock is not too close to the hole one has a plane-wave solution plus some small perturbation W in the region between the two localized objects. Getting closer to the shock the perturbation grows and forms the shock. In order to include the interaction with a shock in the calculations one has to require that W (as introduced in eq.(9)) matches correctly with the shock solution. While up to now, in order to insure global boundedness, we only accepted algebraic growth of W in the inner region, the matching with the shock requires exponential growth. The explicit form of W in the region between hole and shock has to be extracted from the asymptotic form of the shock solution. In general this requires numerical computation since the shocks are not known analytically. If the wave numbers on both sides of the shock are small the shock solution can be approximated by solving the lowest-order phase equation. In the NLS limit the wave numbers selected by the standing hole (and for $\gamma \sim \beta^2 \ll 1$ also by the traveling holes) are of order β, and the corresponding shock solutions can be obtained analytically. Here we consider the case where a standing hole is perturbed by a standing shock. The corresponding phase equation for the shock region reads

$$\partial_t \phi = -\frac{r}{\gamma} + \gamma^{-1}(1 + \beta\gamma)\partial_{xx}\phi + (\frac{\beta}{\gamma} - 1)(\partial_x \phi)^2 \tag{28}$$

which can be solved via a Hopf-Cole transformation. Looking for constantly moving solutions, i.e. $\partial_t \phi = -\Omega = -\omega(q_0)$ where q_0 is the wavenumber of the standing hole (see eq.(B14)), one obtains for the phase gradient

$$\partial_x \phi = q_0 \tanh\left[\frac{p}{2}(x - L)\right] \tag{29}$$

with $p = 2(\beta - \gamma)/(1 + \beta\gamma)q_0$. For $x - L \ll -1$ this yields

$$\partial_x \phi \simeq q_0 \left[1 - 2\exp(-|p|L)e^{|p|x}\right] \tag{30}$$

Here one sees in particular that $W \sim e^{|p|x}$ grows very slowly ($p \sim \beta^2$) compared to the fast decay of the localized mode Ψ_{loc}. Consequently no boundary terms appear in the solvability condition (12) and eq.(17) remains valid in the presence of a shock. However, since the local wave number is perturbed, the phase-conservation condition eq.(18) is altered. In a linear theory the change Δv^s of Δv which results from the presence of the shock can be superposed with the contributions proportional to δk and $\dot{k}$ (in the following we denote them by Δv^0). This yields

$$\Delta v \left[:= v - (1 - \frac{\beta}{\gamma})(q_1 + q_2)\right] = \Delta v^0 - (1 - \frac{\beta}{\gamma})q_2^s \tag{31}$$

where $q_2^s := q_2 - q_0 = -2q_0\exp(-|p|L)$ is the change of q_2 caused by the shock. Δv^0 is given by the rhs of eq.(18). Inserting this into eq.(17) one obtains

$$\dot{v} = \lambda_\delta \left[\delta v + \frac{5}{4}\beta^{-1}\mathcal{R}_\infty^{-2}(\beta - \gamma)^2 q_2^s\right] + O(v, \delta v^3, \delta^2, \beta^3) \tag{32}$$

$$\simeq \lambda_\delta \left[\delta v + \frac{5\sqrt{2}}{6\beta}\mathcal{R}_\infty^{-2}(\beta - \gamma)^3 e^{-|p|L}\right]$$

where λ_δ is given by the rhs of eq.(19). In particular one finds that the hole-shock interaction is always attractive (positive acceleration ; $L > 0$ has been assumed) which is consistent with the results (mostly numerical) presented before (see sections III.D and IV.B of [5]).

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VI. CONCLUDING REMARKS

The above analysis was restricted to the parameter range where the N.B. holes, i.e. the 1-parameter soliton subfamily which survives in the cubic CGLE, is stable or weakly unstable. Here, starting with one of the holes, the system remains for all times close to the 1-parameter subfamily. Thus, in the appropriate parameter range, the present work describes the slowest degree of freedom, which governs the longtime behavior. The problem how an arbitrarily initialized soliton relaxes into this subfamily has not been considered. We also dit not treat the destruction of a N.B. hole (into a plane-wave state) far beyond its stability limit. In these cases Δv does not remain small for all times, i.e. the system is temporarily far away from the N.B. family. Then the relevant time scale should be of order β (or γ) and it is not necessary to include higher-order perturbations to the cubic CGLE. We believe, however, that the method which combines a solvability condition for the core region with a phase-conservation condition obtained from the far field, should in principal be applicable also to more general situations.

APPENDIX A: DESTRUCTION OF THE FAMILY

In the following we derive eq.(13). For this purpose we multiply eq.(8) (with $\partial_t A = 0$) successively by $-iA^*$ and $\psi_{loc}^* := \partial_\zeta A^* + ikA^*$. Defining the "current" $j_{loc} := Re(iA\partial_\zeta \psi_{loc}^*)$ (this yields $j_{loc} = (\partial_\zeta \phi - k)R^2$ if $R(\zeta) := |A(\zeta)| \neq 0$; ϕ refers to the Phase of A) one can write for the real parts of the resulting equations:

$$\partial_\zeta (j_{loc} + QR^2) = \hat{f}(R^2)R^2 - \beta v j_{loc} \tag{A1}$$

$$\partial_\zeta \left(|\psi_{loc}|^2 - \int^{R^2} f(u)du \right) = \hat{f}(R^2)j_{loc} - \beta v |\psi_{loc}|^2 \equiv Re(I[A]\psi_{loc}^*) . \tag{A2}$$

Here $f := \omega - k^2 - R^2$ and $\hat{f} := r - \beta\omega - (\gamma - \beta)R^2 + \delta R^4$ (with $\omega = \Omega + vk$) are polynominals of R^2. For any soliton solution $\mathcal{A}$ which survives the perturbation eq.(12) (with $\partial_t A = 0$) states that the integral over the rhs of eq.(A2) has to be zero if $A = \mathcal{A}$ (then we write $\Psi_{loc}, J_{loc}, \mathcal{R}$ for ψ_{loc}, j_{loc}, R).

In the unperturbed case (i.e. $\beta = \gamma = \delta = r = 0$ so that the rhs of eqs.(A1, A2) vanish) one finds from eq.(A1) that $J_{loc} - Q\mathcal{R}^2$ is a constant. Since J_{loc} vanishes for $\zeta \to \pm\infty$ by construction we can write

$$J_{loc} = Q(\mathcal{R}^2 - \mathcal{R}_\infty^2) . \tag{A3}$$

Similarly one finds from eq.(A2) that $|\Psi_{loc}|^2$ is a second order polynomial of $\mathcal{R}^2$. The double zero for $\zeta = \pm\infty$ allows us to write

$$|\Psi_{loc}|^2 = \frac{1}{2}(\mathcal{R}_\infty^2 - \mathcal{R}^2)^2 . \tag{A4}$$

From eq.(A1) one can further see that also $\hat{f}(\mathcal{R}^2)$ has to vanish asymptotically, i.e.

$$\hat{f}(\mathcal{R}^2) = (\gamma - \beta)(\mathcal{R}_\infty^2 - \mathcal{R}^2) + \delta(\mathcal{R}^4 - \mathcal{R}_\infty^4) \tag{A5}$$

in order to avoid divergencies of j_{loc}. Combining eqs.(A3), (A5) and (A4) we can write the rhs of eq.(A2) at lowest order in β

$$\hat{f}(\mathcal{R}^2)J_{loc} - \beta v|\mathcal{A}_\zeta|^2 = \delta Q(\mathcal{R}^2 - \mathcal{R}_\infty^2)(\mathcal{R}^4 - \mathcal{R}_\infty^4) - \frac{\gamma}{2}\Delta v(\mathcal{R}_\infty^2 - \mathcal{R}^2)^2 \tag{A6}$$

where the rhs can be identified with the lhs of eq.(13) by noting that $Q = \frac{\beta}{\gamma}k + O(\delta)$ holds, as can be seen from eq.(7) and the definition $Q = k - \frac{v}{2}$ (see eq.(3)).

APPENDIX B: PHASE CONSERVATION IN THE TIME DEPENDENT CASE

Here we show how eq.(7) has to be altered when the asymptotic plane-wave states are perturbed. We do this by solving eq.(4) with the ansatz (15) explicitly in the overlap region where $\kappa|\zeta| \gg 1$ (so that hole solutions can be approximated by a plane wave) but $|(\partial_t k)\zeta| \ll 1$ (so that eq.(15) for the time dependence is still valid). Analogously to the derivation of eq.(7) we will require that the solution derived for the regions of overlap holds on both sides independently. As long as both sides are antisymmetric (like for the standing hole) phase conservation is fulfilled trivially. When the symmetry of the standing hole is destroyed phase conservation becomes non trivial and one obtains a relation between the velocity and the symmetry breaking parts of the solution (i.e. k and $\dot{k}$, compare eqs. (7) and (18)).

The following calculation is not restricted to the NLS limit if one generalizes the ansatz (15) by writing

$$\partial_t A = \dot{k}\partial_k \mathcal{A}^{NB}[k(t)] + O(\delta^2) \tag{B1}$$

where $\mathcal{A}^{NB}$ refers to the Nozaki and Bekki solutions [4] (see e.g. Appendix A of [5]). We start from eq.(4) in the moving and constantly rotating frame and separate into real and imaginary part using $A = R\exp[i\phi]$

$$
\begin{aligned}
\dot{R} &= (r - \beta\phi_\zeta^2 - \gamma R^2 + \delta R^4)\ R\ -\ [\phi_{\zeta\zeta}R + 2\phi_\zeta R_\zeta]\ +\ v\ R_\zeta\ +\ \beta\ R_{\zeta\zeta} \\
\dot{\phi}R &= (\Omega - \phi_\zeta^2 - R^2)\ R\ +\ \beta\ [\phi_{\zeta\zeta}R + 2\phi_\zeta R_\zeta]\ +\ v\ \phi_\zeta R\ +\ R_{\zeta\zeta}
\end{aligned}
\tag{B2}
$$

For $\kappa|\zeta| \gg 1$ the Nozaki-Bekki solutions can be approximated by a plane-wave solution (of the CGLE with $\delta = 0$) with some wave number q ($q = q_{2/1}$ for $\pm\kappa\zeta \gg 1$). One has

$$R^2 = \rho_0^2(q) := \frac{1}{\gamma}(r - \beta q^2) \tag{B3}$$

$$\phi = q\zeta + \varphi$$

where φ has to be extracted from the Nozaki-Bekki solutions. In the NLS limit one has $\varphi = \mp\arcsin\sqrt{2}\rho_0^{-1}(\frac{\beta}{\gamma}k - \frac{\Delta v}{2})$ for the left/right overlap region which can be obtained from the soliton solutions (3). Consistently with eq.(15) the time dependence of R and ϕ at lowest order is given by

$$\dot{R} = -\frac{\beta}{\gamma}\frac{q}{\rho_0}\dot{k}\ + O(v^2) \tag{B4}$$

$$\dot{\phi} = (\zeta + \partial_k\varphi)\dot{k}\ + O(v^2) \tag{B5}$$

which corresponds to a change of the wave number ($\dot{q} = \dot{k} + O(v^3)$) plus a change of the phase difference. Since eq.(B5) contains a diverging term the solutions R and ϕ of eqs.(B2) also have to diverge in the overlap region. We thus make the following ansatz:

$$R = \rho + \rho_l\zeta \tag{B6}$$

$$\partial_\zeta\phi = q + q_l\zeta \tag{B7}$$

where ρ_l, q_l vanish for $\dot{k} \to 0$ (we note that $\dot{k} \sim \delta[v + O(v^3)]$). Inserting this into eqs.(B2) one obtains linear terms $\sim \zeta^1$ and constant terms $\sim \zeta^0$ whose coefficients have to cancel respectively. Terms proportional to ζ^2 can be neglected since they are of order higher than $\dot{k}$.

From the linear terms one obtains at order δ^1:

$$\begin{aligned}
0 &= -2\beta q q_l - 2\gamma \rho_0 \rho_l \\
\dot{k} &= -2q q_l - 2\rho_0 \rho_l
\end{aligned} \tag{B8}$$

$$\text{or} \qquad \rho_l = \rho_l(q; \dot{k}) = \frac{k}{2(\frac{\gamma}{\beta}-1)\rho_0} , \qquad q_l = q_l(q; \dot{k}) = \frac{k}{2(\frac{\beta}{\gamma}-1)q} . \tag{B9}$$

Constant terms yield

$$\rho^2 = \rho^2(q; \dot{k}) = \frac{1}{\gamma}\left(r - \beta q^2 + \delta \rho_0^4 - \left[q_l + 2\frac{q\rho_l}{\rho_0}\right] + \frac{\beta}{\gamma}\rho_0^{-2}q\dot{k}\right) + O(v^2) \tag{B10}$$

$$\text{and} \qquad vq = g_{\dot{k}}(q) := -\omega + q^2 + \rho^2 - \beta\left[q_l + 2\frac{q\rho_l}{\rho_0}\right] + \dot{k}\partial_k\varphi + O(v^2) . \tag{B11}$$

In analogy with eq.(7) we now write

$$v = \frac{g_{\dot{k}}(q_1) + g_{\dot{k}}(q_2)}{q_1 - q_2} \tag{B12}$$

which yields

$$\Delta v = -4\delta\frac{\beta}{\gamma^2}\rho_0^2 k + \left[-(1+\beta\gamma)h_s + \frac{\beta}{\gamma^2\rho^2} + \frac{\partial_k\varphi}{q_0}\right]\dot{k} + O(v^2) \tag{B13}$$

$$\text{with} \quad h_s = \frac{\rho_0^2 - 2\frac{\beta}{\gamma}q_0^2}{2(\gamma-\beta)q_0^2\rho_0^2}$$

where $q_0 := q_2(v=0) = -q_1(v=0)$ refers to the wave number of the standing Nozaki-Bekki solution which can be obtained either from these solutions or – in the NLS limit – also by integrating eq.(A1) (with $k = v = 0, \delta = 0$) at order β^1

$$q_0 \simeq \frac{j_{loc}|_{-\infty}^{+\infty}}{2\mathcal{R}_\infty^2} \simeq \frac{1}{2\mathcal{R}_\infty^2}\int_{-\infty}^{+\infty}\hat{f}(\mathcal{R}^2)\mathcal{R}^2 d\zeta = \frac{\sqrt{2}}{3}(\gamma-\beta)\mathcal{R}_\infty \tag{B14}$$

where terms of order β^2 have been neglected.

At the next order the time dependence is given by eq.(20). For the overlap region this yields

$$\dot{R} = -\frac{\beta}{\gamma}\frac{q}{\rho_0}\dot{k} + \frac{1}{2}\dot{\rho_l}\zeta^2 \tag{B15}$$

$$\dot{\phi} = (\zeta + \partial_k\varphi)\dot{k} + \frac{1}{2}\dot{q_l}\zeta^2 + \Delta\dot{v}\partial_{\Delta v}\varphi \tag{B16}$$

where the change of the phase $\partial_{\Delta v}\varphi$ proportional to Δv is only known in the NLS limit. From eqs.(B9) one finds

$$\dot{\rho_l} = \frac{\ddot{k}}{2(\frac{\gamma}{\beta}-1)\rho_0} + O(v^2) \tag{B17}$$

$$\dot{q_l} = \frac{\ddot{k}}{2(\frac{\beta}{\gamma}-1)q} + O(v^2) . \tag{B18}$$

Equations (B2) with (B15) and (B16) can be solved by the ansatz:

$$R = \rho + (\rho_l + \rho_{l2})\zeta + \rho_{ll}\zeta^2 \tag{B19}$$

$$\partial_\zeta\phi = q + (q_l + q_{l2})\zeta + q_{ll}\zeta^2 \tag{B20}$$

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where ρ_{l2}, q_{l2}, ρ_{ll} and q_{ll} are proportional to $\ddot{k}$. (terms containing $\dot{k}^2$ are neglected since they are of order $O(v^2)$.) Collecting terms proportional to ζ^1 and ζ^2 yields at order $\ddot{k}$ four linear equations from which the quantities ρ_{l2}, q_{l2}, ρ_{ll} and q_{ll} can be determined. From the constant terms ($\sim \zeta^0$) one obtains expressions for ρ^2 and for vq which differ from eqs.(B10) and (B11) only by terms proportional to $\ddot{k}$. Then the same procedure as in the last section (see eq.(B12)) leads to eq.(22).

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[1] see e.g.   R.L. Herman J. Phys. A **23**, 2327 (1990)
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[4] N. Bekki and K. Nozaki, Physics Letters **A 110**, 133 (1985).

[5] **previous article**   S. Popp, O. Stiller, I. Aranson and L. Kramer *Holes in the perturbed cubic complex Ginzburg-Landau equation.*

† The notation of the previous article [5] is recovered by rescaling: $b = 1/\beta$, $c = 1/\gamma$ and $d = \delta\gamma/r^2$.